Journal of Engineering Mathematics, Vol. 5, No. 3, July 1971
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

A Non-Linear Theory of Rotating Shallow Shells of Revolution

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(Received May 15, 1970 and in revised form March 2, 1971)

SUMMARY
A non-linear theory for shallow shells of revolution is presented in this paper. While arriving at the differential equations, large deformations and small strains are taken into account. The solution of equations in the edge zones is based on the boundary layer theory. These boundary layer solutions, predominant in the narrow edge zones are superposed on the interior solutions, to obtain a complete solution of the problem. For the interior solution, linear and non-linear theories are worked out and it is found that the agreement between the two theories is good even when the ratio of the transverse displacement to the thickness of the shell is large.

1. Introduction

The problem considered herein is of an axially symmetric shell of arbitrary shape, subjected to the loads due to the spinning of the shell. A non-linear membrane theory of shallow elastic shells has been presented earlier [3]. The solution mentioned in [3], however, does not completely describe the exact behavior of the shell. In the edge zones it is found that the bending terms are important. Because of the large spinning velocity of the shell, it is necessary to retain non-linear terms in the bending theory. The solution is obtained by boundary layer techniques. In the range the boundary layer exists, the solution is comprised of boundary layer solution and interior solution. The former is valid only in narrow edge zones while the later holds for the complete field ($a \leqq r \leqq b$).

It is seen that the non-linear character of the equations depends on a non-dimensional parameter μ defined in equation (v).

The formulation of the problem is based on the Euler–Bernoulli and Kirchoff assumptions. Since the shape of the shell is left arbitrary the governing equations are fairly general in nature. It can be noted that the solution mentioned in [3] is a part of the particular solutions of the differential equations in this paper. Complete solutions are carried out for spherical shells. The numerical computation and the results plotted in the graphs are for spherical shallow shells.

2. Fundamental Equations

Considering the case of large deformations under the homogeneity and isotropy conditions, one can write the strain and displacement relationships for the rotationally symmetric case as follows [3]:

$$\varepsilon_r = U' - Z'w' + \tfrac{1}{2}(w')^2 \tag{2.1a}$$

$$\varepsilon_\theta = U/r \tag{2.1b}$$

where, ε_r and ε_θ are the radial and circumferential strains and u, v, w are the displacements in the directions of r, θ, Z coordinates. Further the moment displacement relations are:

$$M_r = -D(w'' + \nu w'/r) \tag{2.1c}$$

and

$$M_\theta = -D(w'/r + \nu w'') \tag{2.1d}$$

The primes denote the differentiation with respect to r, $D = Eh^3/12(1-\nu^2)$ represents the rigidity of the shell, ν is the Poisson ratio and E is the modulus of elasticity of the material of the shell. Two arbitrary functions ψ and β are introduced which are defined by equation (2.2) in terms of radial force per unit length of sections of the shell, N_r, and displacement w respectively.

$$\psi = rN_r\,, \quad w = -\int \beta\, dr \tag{2.2}$$

The type of shell dealt with in this paper is considered to be mounted around a shaft such that its inner periphery is fixed and the outside is free. Therefore, the boundary conditions at the outside periphery, $r=b$, are written as,

$$M_r(b) = \psi(b) = 0 \tag{2.3}$$

and at the inner boundary $r=a$,

$$\beta(a) = U(a) = 0 \tag{2.4}$$

3. Formulation of Differential Equations

It has been shown elsewhere [1] that the general differential equations of shallow shells of arbitrary shape can be written as,

$$DL\beta + (Z' - \beta)\psi/r = 0$$

and,

$$L\psi/(Eh) - (Z' - \beta/2)\beta/r + (3+\nu)\rho h\Omega^2 r/(Eh) = 0 \tag{3.1}$$

In the above equations, h denotes the thickness of the shell, Ω the spinning velocity of the shell, and ρ the mass density of the material of the shell. Further, while writing the above equations, quantities of the order of ϕ^3, $\phi^2\beta$, $\phi\beta^2$ etc. have been neglected. In equation (3.1) the equi-dimensional operator L is given by the expression,

$$L = d^2/dr^2 + (1/r)d/dr - 1/dr^2 \tag{i}$$

The particular solution of equations (3.1) is given as

$$\psi_1 = 0$$

and

$$\beta_1 = Z' \pm [(Z')^2 - 2(3+\nu)\rho h\Omega^2 r^2/(Eh)]^{\frac{1}{2}} \tag{3.2}$$

Equations (3.2) essentially are two solutions but because of the physical conditions of the problem, discontinuities are disallowed and the solution with a negative sign in (3.2b) should be adopted. Substituting equations (3.2) in (3.1) and writing the variables β and ψ in two parts as

$$\beta = \beta_1 + \beta^*$$

and

$$\psi = 0 + \psi^*\,, \tag{ii}$$

results in eqs. (3.3):

$$DL\beta^* + (Z' - \beta_1)\psi^*/r - \beta^*\psi^*/r = 0$$

$$L\psi^*/(Eh) - (Z' - \beta_1)\beta^*/r + \tfrac{1}{2}\beta^{*2}/r = 0 \tag{3.3}$$

Equations (3.3) are in terms of the modified functions β^* and ψ^* which are related to original β

and ψ by eq. (ii). Here in what follows, a general theory for boundary layer occurring near the inner periphery is developed. The same technique is used for working out the boundary layer solution near the outer periphery. Before developing a solution of equation (3.3) various quantities involved are non-dimensionalized by writing.

$$x = (r-a)/l$$

and

$$\beta^* \to \beta_D \beta\,, \qquad \beta_1 \to \beta_{1D} \tilde{\beta}\,,$$
$$Z_{,x} \to Z_D \tilde{Z}_{,x}\,, \qquad \psi^* \to \psi_D \psi \tag{3.4}$$

Here, $Z_{,x}$ represents the differential dZ/dx. The parameter l denotes the boundary layer width and is a small quantity. The quantities β_D, β_{1D}, Z_D and ψ_D are introduced such that the variables β, $\tilde{\beta}$, $\tilde{Z}_{,x}$, and ψ are completely non-dimensionalized. A further discussion on how these quantities can be fixed will be given later in this paper. By introducing equations (3.4) in (3.3), the basic differential equations can be reduced in terms of the non-dimensional parameters x, β, $\tilde{\beta}$, $\tilde{Z}_{,x}$, and ψ.

$$D/l^2 \beta_D L^* \beta + Z_D \tilde{Z}_{,x}[1 - l/(Z_D \tilde{Z}_{,x})\beta_{1D}\tilde{\beta}]\psi_D \psi/[(1+lx/a)al]$$
$$-\beta_D \psi_D \beta\psi/(a+lx) = 0$$
$$\psi_D/(l^2 Eh) L^* \psi - Z_D \tilde{Z}_{,x}[1 - l/(Z_D \tilde{Z}_{,x})\beta_{1D}\tilde{\beta}]\beta_D \beta/[(1-lx/a)al]$$
$$+ (\tfrac{1}{2})\beta_D^2 \beta^2/(a+lx) = 0 \tag{3.5}$$

The non-dimensional form of the equidimensional operator L is given by L^* in equation (iii)

$$L^* = d^2/dx^2 - l/(a-lx)d/dx - [l/(a-lx)]^2 \tag{iii}$$

Until this point, the parameters β_D, ψ_D and l have been left arbitrary. The parameters β_{1D} and Z_D are known if the shape of the shell is defined. Since the solution is being sought in the boundary layer zone which occurs if the quantity $l/(b-2) \ll 1$, it is justifiable to write, $\beta/Z_{,x} = O(1)$. A relationship between β_D, ψ_D and l is established by making the first two terms of equation (3.5) of the same order.

$$\psi_D/(l^2 Eh) = Z_D(1 - \beta_{1D}/Z_D)\beta_D/(al) \tag{3.6}$$
$$D/l^2 \beta_D = Z_D(1 - l\beta_{1D}/Z_D)\psi_D/(al)$$

Making use of equations (3.4) in (3.2) and simplifying the following equation is obtained.

$$\beta_1 = Z_D/l\tilde{Z}_{,x}\{1 \pm [1 - \{(1+l/ax)/\tilde{Z}_{,x}\}^2 \Omega']^{\frac{1}{2}}\} \tag{3.7}$$

The non-dimensional velocity parameter Ω' is introduced for the simplification of equations and is defined as

$$\Omega' = 2(3+\nu)a^2 l^2 \rho h \Omega^2/(Z_D^2 Eh) \tag{iv}$$

It can be seen that the solution of the form of (3.7) is possible if

$$\Omega' < \min.[\tilde{Z}_{,x}/(1+l/ax)]^2 \qquad 0 < x < (b-a)/l \tag{3.8}$$

4. Boundary Conditions

The boundary conditions (2.3) and (2.4) when expressed in terms of the functions β and ψ and then written in terms of perturbation and non-dimensional parameters according to (3.4), yield,

$$\beta_D/l\{d\beta/dx|_{x=(b-a)/l} + \nu l/b\beta|_{x=(b-a)/l}\} = -\beta_1'(b) - \nu/b\beta_1(b)$$
$$\psi(b) = 0 \tag{4.1}$$

and

$$\beta_1(a)+\beta_D\beta(0)=0$$
$$a/l\, d\psi/dx|_{x=0}-\nu\psi(0)=-\rho h\Omega^2 a^3/\psi_D$$

The two linear terms in (4.1c) will be of the same order if,

$$\beta_D=-\frac{b}{a}\beta_1(a) \tag{4.2}$$

Equation (4.2) along with (3.6) completely determine the three parameters β_D, ψ_D and l.

$$\psi_D=-Z_D b/a^2(1-l/Z_D\beta_{1D})Ehl\beta_1(a)$$
$$l\ \ =1/\sqrt{12(1-\nu^2)}(ha/Z_D)1/(1-l/Z_D\beta_{1D}) \tag{4.3}$$

Using equations (3.6) and (4.2), the governing non-linear differential equations (3.5) yield,

$$L^*\beta+\tilde{Z},x/(1+l/ax)\psi+(\mu b/a)\beta\psi/(1+l/ax)=0$$
$$L^*\psi-\tilde{Z},x/(1+l/ax)\beta-\tfrac{1}{2}(\mu b/a)\beta^2/(1+l/ax)=0 \tag{4.4}$$

where the non-linearity parameter μ is given as,

$$\mu=\beta_1(a)l/[Z_D(1-l/Z_D\beta_{1D})] \tag{v}$$

5. Solution of the equations

At this stage the problem can be divided in two parts:
(i) when the parameter $\mu\ll 1$, then all the differential equations can be linearized.

$$L^*\beta+\tilde{Z}_{,x}/(1+l/ax)\psi=0$$
$$L^*\psi-Z_{,x}/(1+l/ax)\beta=0 \tag{5.1}$$

(ii) When the quantity $l/(b-a)\ll 1$, the existence of the boundary layer is guaranteed, which is not the case in (i). For the case when $l/a\ll 1$, the differential equations, thus simplify to

$$d^2\beta/dx^2+\tilde{Z}_{,x}\psi=0$$
$$d^2\psi/dx^2-\tilde{Z}_{,x}\beta=0 \tag{5.2}$$

6. Example of a spherical cap

(i) Boundary layer solution near the inner edge: For the case of a shallow spherical shell, the particular solution given by equation (3.2) takes the following form,

$$\beta_1=\kappa r$$

where,

$$\kappa=1/R\pm[1/R^2-2(3+\nu)\rho\Omega^2/E]^{\frac{1}{2}} \tag{6.1}$$

Various other parameters introduced in the general theory reduce to,

$$\beta_D=-\kappa b$$
$$l=[12(1-\nu^2)]^{-\frac{1}{4}}\{h/(1/R-\kappa)\}^{\frac{1}{2}}$$
$$\psi_D=-[12(1-\nu^2)]^{-\frac{1}{2}}Eh^2\kappa b$$

and

$$\mu = R\kappa/(1-R\kappa) \tag{6.2}$$

The differential equations (5.1) and (5.2) yield,

$$L^*\beta+\psi=0$$
$$L^*\psi-\beta=0 \tag{6.3a}$$

and

$$d^2\beta/dx^2+\psi=0$$
$$d^2\psi/dx^2-\beta=0 \tag{6.3b}$$

Combining the two functions β and ψ and introducing a complex function F such that

$$F=\beta+i\psi\ ,$$

the two equations (6.3a) can be written in a convenient form

$$L^*F-iF=0$$

Where i is the complex number $\sqrt{-1}$. The solution of this equation is written as,

$$F=\beta+i\psi=C_1' I_1[\sqrt{i}\,(a/l+x)]+C_2' K_1[\sqrt{i}\,(a/l+x)]$$

I_1 and K_1 being the associated Bessel functions and C_1' and C_2' are complex constants. Expressing I_1 and K_1 in terms of Kelvin and Thompson function, the solution for β and ψ can be easily obtained.

$$\beta=C_1\ \mathrm{ber}_1(a/l+x)-C_2\ \mathrm{bei}_1(a/l+x)+C_3\ \mathrm{Ker}_1(a/l+x)$$
$$-C_4\ \mathrm{Kei}_1(a/l+x) \tag{6.4}$$

$$\psi=C_1\ \mathrm{bei}_1(a/l+x)+C_2\ \mathrm{bei}_1(a/l+x)+C_3\ \mathrm{Kei}_1(a/l+x)$$
$$+C_4\ \mathrm{Ker}_1(a/l+x) \tag{6.4}$$

Similarly equations (6.3b) can be combined together which will then yield

$$\psi=\exp(-x/\sqrt{2})\,C_1^*\ \cos(x/\sqrt{2})-a/b\ \sin(x/\sqrt{2})$$
$$\beta=\exp(-x/\sqrt{2})\,C_1^*\ \sin(x/\sqrt{2})+a/b\ \cos(x/\sqrt{2}) \tag{6.5}$$

where

$$C_1^*=-[1-\sqrt{2}\,\rho h\Omega^2a^2l/\psi_D]/(1-\sqrt{2}\nu\, l/a)$$

(ii) Boundary layer solution near the outer edge: For this case the non-dimensionalization of the variable x is achieved by setting

$$x=(b-r)/l$$

Proceeding in the same manner as before, one arrives at a solution similar to (6.5)

$$\psi=C_1^*\ \sin(x/\sqrt{2})\exp(-x/\sqrt{2}) \tag{6.6}$$
$$\beta=C_1^*\ \cos(x/\sqrt{2})\exp(-x/\sqrt{2})$$

In this case the constant C_1^* is given as

$$C_1^*=\sqrt{2}(1+\nu)/(\sqrt{2}+b/l) \tag{6.6}$$

It can be seen that if $a\to 0$, there is no boundary layer at the inner periphery. It only exists on the outside periphery. The complete boundary layer solution for this case is obtained by making use of equations (6.6), (3.4), (2.2), and (ii). After simplification, the solution gives,

$$\psi = \psi_D C_1^* \sin(x/\sqrt{2}) \exp(-x/\sqrt{2})$$
$$\beta = \kappa[(b-lx) - bC_1^* \cos(x/\sqrt{2}) \exp(-x/\sqrt{2})]$$
$$N_r = \psi/(b-lx)$$
$$N_\theta = \psi_D C_1^*/l \{\cos(x/\sqrt{2}) - \sin(x/\sqrt{2}) \exp(-x/\sqrt{2}) - -\tfrac{1}{2}\Omega'/(3-v)[(b-lx)/R]^2\}$$

and

$$w = l\kappa/\sqrt{2}\{bx - \tfrac{1}{2}lx^2 - b/\sqrt{2}C_1^*[\sin(x/\sqrt{2}) - \cos(x/\sqrt{2})]\exp(-x/\sqrt{2})\} - -l\kappa/\sqrt{2}\{b(b-a)/l - \tfrac{1}{2}(b-a)^2/l - -b/\sqrt{2}C_1^*[\sin((b-a)(\sqrt{2})) - \cos(b-a)/(\sqrt{2}l)]\exp(-(b-a)/(\sqrt{2}l))\} \tag{6.7}$$

(iii) Second Particular solution of (3.1). For the spherical case the second solution of equations (3.1) can be written as,

$$\beta = r/R$$
$$\psi = C_1 r + C/r + (\tfrac{1}{16}) Ehr^3(1-\Omega')R^2 \tag{vi}$$

Equations (vi) are written for $\Omega' \geqq 1$. For the case of $\Omega' = 1$, the shell flattens out to a flat plate and $\psi = 0$. The complete solution for $\Omega' > 1$ will not be carried out here.

(iv) Interior Solution: Linear membrane solution of equations (3.1) is given as

$$\psi = 0 \quad \text{and} \quad \beta = \tfrac{1}{2}\Omega' r/R$$

With these values of ψ and β, the expressions for the stresses and deflection are,

$$N_r = 0\,, \qquad N_\theta = \rho h \Omega^2 r^2$$

and

$$w = -\tfrac{1}{4}\Omega'(r^2 - a^2)/R \tag{6.8}$$

The non-linear membrane solution is the same as in (3.2) which yield

$$N_r = 0\,, \qquad N_\theta = \tfrac{1}{2}\Omega'/(3+v)(r/R)^2 \tag{6.9}$$

and

$$w = \tfrac{1}{2}[1 - (1-\Omega')^{\frac{1}{2}}](r^2 - a^2)/R$$

For the interior solution, valid in the zone $a \leqq r \leqq b$, the problem could also be considered as a linear bending problem. For this case, equations (3.1) are linearized and for obtaining their homogeneous solution, the equations are combined together in the form

$$L(L\beta) + \delta^4 \beta = 0$$

where

$$\delta = [12(1-v^2)/(h^2 R^2)]^{\frac{1}{4}} \tag{vii}$$

The solution of this equation can be easily written as

$$\beta = AJ_1(\sqrt{i}\,\delta r) + BI_1(\sqrt{i}\,\delta r)$$

While arriving at this solution the other two constants are set to zero due to the improper behavior of Y_1 and K_1 at $r = 0$. Superimposing the particular solution and applying the boundary condition along with the finiteness condition at $r = 0$, the solution can be simplified to yield,

$$\beta = (C_1 + C_4)\,\mathrm{ber}_1(\delta r) + (C_2 + C_3)\,\mathrm{bei}_1(\delta r) + \tfrac{1}{2}\Omega' r/R$$
$$\psi = RD\delta^2[(C_1 - C_4)\,\mathrm{bei}_1(\delta r) - (C_2 - C_3)\,\mathrm{ber}_1(\delta r)] \tag{6.10}$$

In equation (6.10) the constants $C_1 \ldots C_4$ are given as below while the constant δ is given by equation (vii).

$$C_1 = (-1)\frac{\dfrac{(1-\nu^2)}{\delta^2 bR}\,\Omega'\eta\,\mathrm{ber}_1(\delta b) + \dfrac{(1+\nu)}{\sqrt{2\delta R}}\,\Omega'\zeta\,\mathrm{bei}_1(\delta b)}{\dfrac{4(1-\nu)^2}{\delta^2 b^2}\,\eta^2 + 2\zeta^2} \tag{6.11a}$$

$$C_2 = (-1)\frac{\dfrac{(1-\nu^2)}{\delta^2 bR}\,\Omega'\eta\,\mathrm{bei}_1(\delta b) - \dfrac{(1+\nu)}{\sqrt{2\delta R}}\,\Omega'\zeta\,\mathrm{ber}_1(\delta b)}{\dfrac{4(1-\nu)^2}{\delta^2 b^2}\,\Omega^2 - 2\zeta^2} \tag{6.11b}$$

$$C_3 = (-1)\frac{\dfrac{(1-\nu^2)}{\delta^2 bR}\,\Omega'\eta\,\mathrm{ber}_1(\delta b) - \dfrac{(1+\nu)}{\sqrt{2\delta R}}\,\zeta\,\mathrm{bei}_1(\delta b)}{\dfrac{4(1-\nu)^2}{\delta^2 b^2}\,\eta^2 + 2\zeta^2} \tag{6.11c}$$

$$C_4 = \frac{\dfrac{(1-\nu^2)}{\delta^2 bR}\,\Omega'\eta\,\mathrm{bei}_1(\delta b) - \dfrac{(1+\nu)}{\sqrt{2\delta R}}\,\Omega'\zeta\,\mathrm{ber}_1(\delta b)}{\dfrac{4(1-\nu)^2}{\delta^2 b^2}\,\eta^2 + 2\zeta^2} \tag{6.11d}$$

$$\zeta = \mathrm{ber}_1(\delta b)\,\mathrm{ber}(\delta b) + \mathrm{bei}_1(\delta b)\,\mathrm{bei}(\delta b) + \mathrm{bei}_1(\delta b)\,\mathrm{ber}(\delta b) - \mathrm{ber}_1(\delta b)\,\mathrm{bei}(\delta b)$$
$$\eta = [\mathrm{ber}_1(\delta b)]^2 + [\mathrm{bei}_1(\delta b)]^2$$

The expressions for stresses and deformation for this case. ($a=0$) can now be obtained from eqs. (6.10)

$$N_r = -D\delta^2 R/r[(C_1 - C_4)\,\mathrm{bei}_1(\delta r) - (C_2 - C_3)\,\mathrm{ber}_1(\delta r)]$$
$$N_\theta = D\delta^2 R/r[(C_2 - C_3)\,\mathrm{ber}_1(\delta r) - (C_1 - C_4)\,\mathrm{bei}_1(\delta r)] + D\delta^3 R/\sqrt{2}[(C_1 - C_2 + C_3 - C_4)\,\mathrm{ber}_1(\delta r) - (C_1 - C_2 - C_3 - C_4)\,\mathrm{bei}(\delta r)] + \rho h\Omega^2 r^2$$

and

$$w = [(C_1 + C_2 + C_3 + C_4)\,\mathrm{ber}_1(\delta r) - (C_1 - C_2 - C_3 + C_4)\,\mathrm{bei}_1(\delta r)]/(2\delta^2)^{\frac{1}{2}} - \tfrac{1}{4}\Omega' r^2/R + (C_1 + C_2 + C_3 + C_4)/(2\delta^2)^{\frac{1}{2}}$$

7. Results and conclusions

Detailed computations have been done for the case of a complete shell ($a=0$). In the computations of the interior solution the results from the linear approximation are comparable to those obtained from linear and non-linear membrane theories. It was found that for $\Omega' = \frac{1}{4}$ the difference in the deflection obtained from linear approximation and linear membrane theory is less than 1%. The deflections obtained from non-linear membrane theory are about 5% higher than the linear approximation. It is interesting to note that for $\Omega' = \frac{1}{4}$ the agreement between the linear and non-linear membrane theories mentioned in the interior solution is

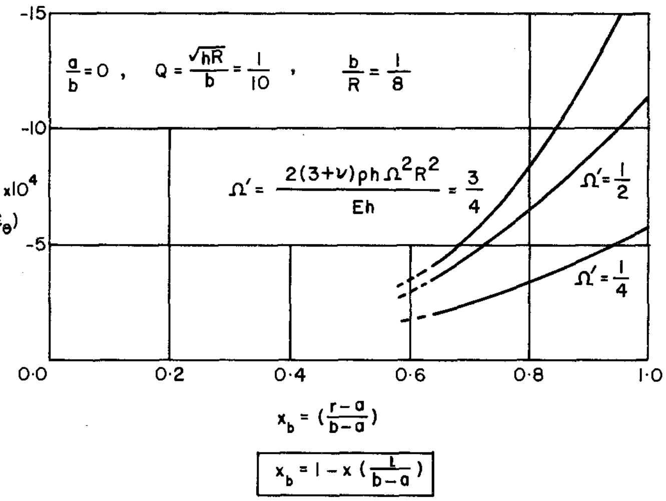

Figure 1. Boundary layer solution for outer boundary.

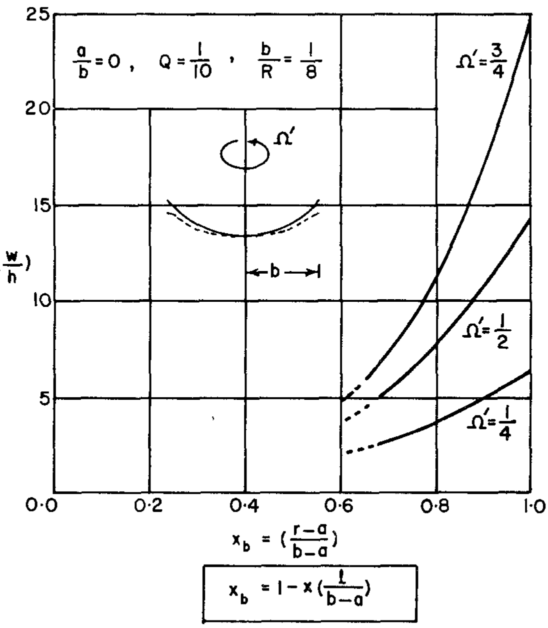

Figure 2. Boundary layer solution for outer boundary.

good even for large deformations ($w/h = 6$).

As it has been discussed earlier, the boundary layer exists and the deviations from the linear and non-linear membrane solutions are confined to narrow edge zones if the following inequality holds.

$$l/(b-a) = (1/b)\sqrt{hR}\,\{1/[12(1-\Omega')(1-\nu^2)]^{\frac{1}{4}}\}/(1-a/b) \ll 1\,. \tag{7.1}$$

If the shell is not very thin and not too shallow, the above condition is satisfied if Ω' is small. The non-linearity parameter μ can be written as

$$\mu = [1-(1-\Omega')^{\frac{1}{2}}]/(1-\Omega')^{\frac{1}{2}} \tag{7.2}$$

It can easily be noted that for the cases discussed above, if the inequality (7.1) is satisfied,

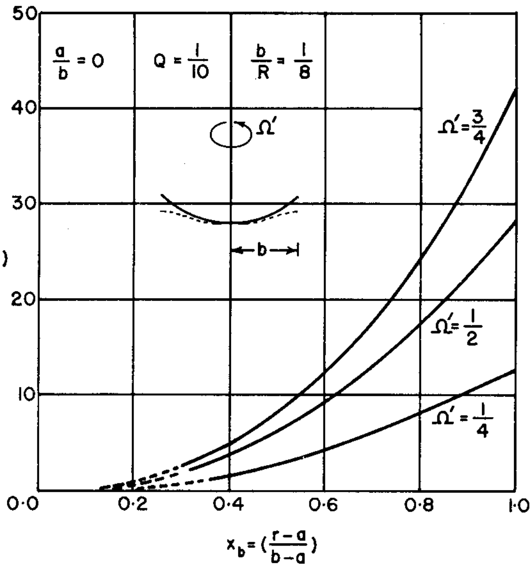

Figure 3. Superposition of boundary layer solution and interior solution.

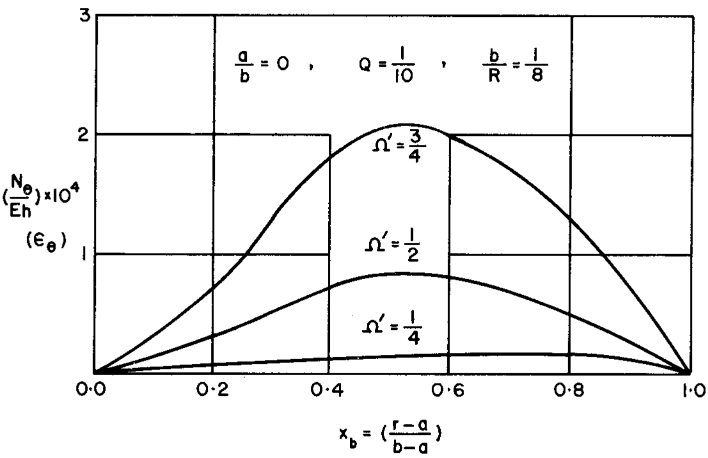

Figure 4. Superposition of boundary layer solution and interior solution.

μ is always small and under these conditions the differential equations can always be linearized.

An upper bound on the deflection w can be prescribed if $\mu \ll 1$ we know that, $w \leqq 0(\Omega' b^2/R)$. Thus the condition $\mu \ll 1$, prescribes that,

$$|w| \ll \frac{b^2}{R}$$

Acknowledgements

The author would like to express his thanks to Professor M. W. Johnson of the University of Wisconsin whose suggestions have been very useful in the preparation of this report.

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Journal of Engineering Math., Vol. 5 (1971) 161–170